Unsupervised Articulated Skeleton Extraction from Point Set Sequences Captured by a Single Depth Camera

Supplemental Derivation Material

[ ]

Abstract

This document shows the derivation procedure of node transformations optimization (Section 4 in our paper) and bone transformations optimization (Section 6 in our paper). The variables in our derivation in this supplemental document are consistent with our paper. In Table 1, we list the general variables which are used in both the optimizations of node transformations and bone transformations.

Table 1: General variables in our paper.

Variables	Meaning	Variables	Meaning
t	the t-th frame	$\mathbf{v}_i^t$	the i-th original point (i.e., position) in frame t
N^t	number of the original points in frame t	$\mathbf{y}_m^t$	the m-th registered point in frame t
$\mathbf{q}_m$	the m-th point in the rest pose	$\mathbf{x}_m^t$	the m-th deformed point of $\mathbf{q}_m$ in frame t
M	number of points in $\{\mathbf{q}_m\}$	D	the dimension (i.e., 3) of the point clouds
$\mathbf{V}^t$	$\mathbf{V}^t = \{\mathbf{v}_i^t\}$, $D \times N^t$ matrix	$\mathbf{Y}^t$	$\mathbf{Y}^t = \{\mathbf{y}_m^t\}$, $D \times M$ matrix
$\mathbf{Q}$	$\mathbf{Q} = \{\mathbf{q}_m\}$, $D \times M$ matrix	$\mathbf{X}^t$	$\mathbf{X}^t = \{\mathbf{x}_m^t\}$, $D \times M$ matrix

1 Node Transformations Optimization

Eq. (1) represents the total energy in the M-step of the Non-rigid Registration (Section 4 in our paper). To be clear, the key variables involved in node transformations optimization are presented in Table 2. *Note that we omit the frame number t here for simplicity.*

Table 2: Key variables involved in optimization of node transformations.

Variables	Meaning	Variables	Meaning
β_{smooth}	weight of the smooth term	β_{small}	weight of the small motion term
$\mathbf{y}'_m$	new position induced by its neighboring nodes	$\mathbf{R}_j$	the j-th node rotation
$\mathbf{T}_j$	the j-th node translation	$\mathbf{n}_j$	the j-th node (i.e., position)
$\mathbf{n}_k$	$\mathbf{n}_k \in N(\mathbf{n}_j)$, i.e., neighbors of $\mathbf{n}_j$	$\mathbf{R}_j^{pre}\vert\mathbf{T}_j^{pre}$	the j-th node transformation at the previous iteration

$$
\begin{aligned}
E =& \frac{1}{2\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\|\mathbf{v}_i - \mathbf{y}'_m\|^2 + \frac{DN_p}{2}\log\sigma^2 \\
&+ \frac{\beta_{\text{smooth}}}{2}\sum_{\mathbf{n}_j}\sum_{\mathbf{n}_k\in N(\mathbf{n}_j)} \bar{\omega}_{jk}\|\mathbf{R}_j(\mathbf{n}_k - \mathbf{n}_j) + \mathbf{n}_j + \mathbf{T}_j - (\mathbf{n}_k + \mathbf{T}_k)\|^2 \\
&+ \frac{\beta_{\text{small}}}{2}\sum_{j}\|\mathbf{R}_j - \mathbf{R}_j^{pre}\|_F^2 + \|\mathbf{T}_j - \mathbf{T}_j^{pre}\|^2
\end{aligned} \tag{1}
$$

Here, $\mathbf{y}'_m = \sum_{\mathbf{n}_j}\bar{\omega}_{mj}[\mathbf{R}_j(\mathbf{y}_m - \mathbf{n}_j) + \mathbf{n}_j + \mathbf{T}_j]$, $\bar{\omega}_{mj} = \bar{\omega}(\mathbf{y}_m, \mathbf{n}_j, r_j)$ and $\bar{\omega}_{jk} = 1$.

1.1 Partial Derivative with Respect to $\mathbf{T}_{\hat{j}}$

We first take the partial derivative of E with respect to $\mathbf{T}_{\hat{j}}$ of a *specific node* $\hat{j}$ and equate it to zero.

$$\begin{aligned}\frac{\partial \mathbb{E}}{\partial \mathbf{T}_{\hat{j}}} =& -\frac{2}{2\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\bar{\omega}_{m\hat{j}}(\mathbf{v}_i - \mathbf{a}_{m\hat{j}} - \bar{\omega}_{m\hat{j}}[\mathbf{R}_{\hat{j}}(\mathbf{y}_m - \mathbf{n}_{\hat{j}}) + \mathbf{n}_{\hat{j}} + \mathbf{T}_{\hat{j}}]) \\ &+ \frac{2\beta_{\text{smooth}}}{2}\sum_{\mathbf{n}_k \in N(\mathbf{n}_{\hat{j}})} [\mathbf{R}_{\hat{j}}(\mathbf{n}_k - \mathbf{n}_{\hat{j}}) + \mathbf{n}_{\hat{j}} + \mathbf{T}_{\hat{j}} - (\mathbf{n}_k + \mathbf{T}_k)] - [\mathbf{R}_k(\mathbf{n}_{\hat{j}} - \mathbf{n}_k) + \mathbf{n}_k + \mathbf{T}_k - (\mathbf{n}_{\hat{j}} + \mathbf{T}_{\hat{j}})] \\ &+ \frac{2\beta_{\text{small}}}{2}(\mathbf{T}_{\hat{j}} - \mathbf{T}_{\hat{j}}^{pre}) = \mathbf{0}\end{aligned} \tag{2}$$

Here, $a_{m\hat{j}} = \sum_{\mathbf{n}_j \neq \mathbf{n}_{\hat{j}}} \bar{\omega}_{mj}[\mathbf{R}_j(\mathbf{y}_m - \mathbf{n}_j) + \mathbf{n}_j + \mathbf{T}_j]$. Reorganizing Eq. (2), we can obtain

$$\begin{aligned}&\mathbf{T}_{\hat{j}}\Big(\frac{1}{\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\bar{\omega}_{m\hat{j}}^2 + 2\beta_{\text{smooth}}\sum_{\mathbf{n}_k \in N(\mathbf{n}_{\hat{j}})} 1 + \beta_{\text{small}}\Big) + \mathbf{R}_{\hat{j}}\Big(\frac{1}{\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\bar{\omega}_{m\hat{j}}^2(\mathbf{y}_m - \mathbf{n}_{\hat{j}}) + \beta_{\text{smooth}}\sum_{\mathbf{n}_k \in N(\mathbf{n}_{\hat{j}})}(\mathbf{n}_k - \mathbf{n}_{\hat{j}})\Big) \\ &= -\Big\{-\frac{1}{\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\bar{\omega}_{m\hat{j}}(\mathbf{v}_i - \mathbf{a}_{m\hat{j}} - \bar{\omega}_{m\hat{j}}\mathbf{n}_{\hat{j}}) + \beta_{\text{smooth}}\sum_{\mathbf{n}_k \in N(\mathbf{n}_{\hat{j}})}[2(\mathbf{n}_{\hat{j}} - \mathbf{n}_k - \mathbf{T}_k) - \mathbf{R}_k(\mathbf{n}_{\hat{j}} - \mathbf{n}_k)] - \beta_{\text{small}}\mathbf{T}_{\hat{j}}^{pre}\Big\}\end{aligned} \tag{3}$$

Let $C_{\hat{j}} = \frac{1}{\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\bar{\omega}_{m\hat{j}}^2 + 2\beta_{\text{smooth}}\sum_{\mathbf{n}_k \in N(\mathbf{n}_{\hat{j}})} 1 + \beta_{\text{small}}$, $\mu_y = \frac{1}{C_{\hat{j}}}\Big(\frac{1}{\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\bar{\omega}_{m\hat{j}}^2(\mathbf{y}_m - \mathbf{n}_{\hat{j}}) + \beta_{\text{smooth}}\sum_{\mathbf{n}_k \in N(\mathbf{n}_{\hat{j}})}(\mathbf{n}_k - \mathbf{n}_{\hat{j}})\Big)$ and $\mu_v = -\frac{1}{C_{\hat{j}}}\{-\frac{1}{\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\bar{\omega}_{m\hat{j}}(\mathbf{v}_i - \mathbf{a}_{m\hat{j}} - \bar{\omega}_{m\hat{j}}\mathbf{n}_{\hat{j}}) + \beta_{\text{smooth}}\sum_{\mathbf{n}_k \in N(\mathbf{n}_{\hat{j}})}[2(\mathbf{n}_{\hat{j}} - \mathbf{n}_k - \mathbf{T}_k) - \mathbf{R}_k(\mathbf{n}_{\hat{j}} - \mathbf{n}_k)] - \beta_{\text{small}}\mathbf{T}_{\hat{j}}^{pre}\}$. Then we can obtain:

$$\mathbf{T}_{\hat{j}} = \mu_v - \mathbf{R}_{\hat{j}}\mu_y \tag{4}$$

$C_{\hat{j}}$, μ_v and μ_y can be further rewritten in matrix form. $C_{\hat{j}} = \frac{1}{\sigma^2}(\mathbf{P1})^T\mathbf{\Omega}_{:,\hat{j}}^2 + 2\beta_{\text{smooth}}|N(\mathbf{n}_{\hat{j}})| + \beta_{\text{small}}$, $\mu_v = -\frac{1}{C_{\hat{j}}}\{-\frac{1}{\sigma^2}[(\mathbf{PX}^T)^T\mathbf{\Omega}_{:,\hat{j}} - \mathbf{A}_{\hat{j}}diag(\mathbf{\Omega}_{:,\hat{j}})\mathbf{P1} - \mathbf{n}_{\hat{j}}(\mathbf{P1})^T\mathbf{\Omega}_{:,\hat{j}}^2] + \beta_{\text{smooth}}\sum_{\mathbf{n}_k \in N(\mathbf{n}_{\hat{j}})}[2(\mathbf{n}_{\hat{j}} - \mathbf{n}_k - \mathbf{T}_k) - \mathbf{R}_k(\mathbf{n}_{\hat{j}} - \mathbf{n}_k)] - \beta_{\text{small}}\mathbf{T}_{\hat{j}}^{pre}\}$, $\mu_y = \frac{1}{C_{\hat{j}}}[\frac{1}{\sigma^2}(\mathbf{Y}diag(\mathbf{\Omega}_{:,\hat{j}}^2)\mathbf{P1} - \mathbf{n}_{\hat{j}}(\mathbf{P1})^T\mathbf{\Omega}_{:,\hat{j}}^2) + \beta_{\text{smooth}}\sum_{\mathbf{n}_k \in N(\mathbf{n}_{\hat{j}})}(\mathbf{n}_k - \mathbf{n}_{\hat{j}})]$. $\mathbf{A}_{\hat{j}}$ is a $D \times M$ matrix with elements $a_{m\hat{j}} = \sum_{\mathbf{n}_j \neq \mathbf{n}_{\hat{j}}} \bar{\omega}_{mj}[\mathbf{R}_j(\mathbf{y}_m - \mathbf{n}_j) + \mathbf{n}_j + \mathbf{T}_j]$. $\mathbf{1}$ is a $N \times 1$ vector with all ones. $\mathbf{\Omega}$ is a $M \times K$ matrix with elements $\bar{\omega}_{mj}$ and $\mathbf{\Omega}_{:,\hat{j}}$ is the $\hat{j}$-th column of $\mathbf{\Omega}$. $\mathbf{\Omega}_{:,\hat{j}}^2$ is the column consisting of squared values of elements in $\mathbf{\Omega}_{:,\hat{j}}$. $diag()$ denotes the diagonal matrix. $\mathbf{P} = \{p_{mi}\}$.

1.2 Solving Node Transformations

We can obtain the following function involving only $\mathbf{R}_{\hat{j}}$ by substituting Eq. (4) into Eq. (1).

$$\begin{aligned}E(\mathbf{R}_{\hat{j}}) =& \frac{1}{2\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\|\mathbf{v}_i - a_{m\hat{j}} - \bar{\omega}_{m\hat{j}}[\mathbf{R}_{\hat{j}}(\mathbf{y}_m - \mathbf{n}_{\hat{j}}) + \mathbf{n}_{\hat{j}} + \mathbf{T}_{\hat{j}}]\|^2 \\ &+ \frac{\beta_{\text{smooth}}}{2}\sum_{\mathbf{n}_k \in N(\mathbf{n}_{\hat{j}})} \|\mathbf{R}_{\hat{j}}(\mathbf{n}_k - \mathbf{n}_{\hat{j}}) + \mathbf{n}_{\hat{j}} + \mathbf{T}_{\hat{j}} - (\mathbf{n}_k + \mathbf{T}_k)\|^2 + \|\mathbf{R}_k(\mathbf{n}_{\hat{j}} - \mathbf{n}_k) + \mathbf{n}_k + \mathbf{T}_k - (\mathbf{n}_{\hat{j}} + \mathbf{T}_{\hat{j}})\|^2 \\ &+ \frac{\beta_{\text{small}}}{2}\Big(\|\mathbf{R}_{\hat{j}} - \mathbf{R}_{\hat{j}}^{pre}\|_F^2 + \|\mathbf{T}_{\hat{j}} - \mathbf{T}_{\hat{j}}^{pre}\|^2\Big)\end{aligned} \tag{5}$$

We denote the first term in the right side of Eq. (5) as E_1.

$$
\begin{aligned}
E_1 &= \frac{1}{2\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\|\mathbf{v}_i - a_{m\hat{j}} - \bar{\omega}_{m\hat{j}}[\mathbf{R}_{\hat{j}}(\mathbf{y}_m - \mathbf{n}_{\hat{j}}) + \mathbf{n}_{\hat{j}} + \mathbf{T}_{\hat{j}}]\|^2 \\
&= \frac{1}{2\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\|\mathbf{v}_i - a_{m\hat{j}} - \bar{\omega}_{m\hat{j}}[\mathbf{R}_{\hat{j}}(\mathbf{y}_m - \mathbf{n}_{\hat{j}}) + \mathbf{n}_{\hat{j}} + \mu_v - \mathbf{R}_{\hat{j}}\mu_y]\|^2 \\
&= \frac{1}{2\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\|(\mathbf{v}_i - a_{m\hat{j}} - \bar{\omega}_{m\hat{j}}\mathbf{n}_{\hat{j}} - \bar{\omega}_{m\hat{j}}\mu_v) - \mathbf{R}_{\hat{j}}(\bar{\omega}_{m\hat{j}}\mathbf{y}_m - \bar{\omega}_{m\hat{j}}\mathbf{n}_{\hat{j}} - \bar{\omega}_{m\hat{j}}\mu_y)\|^2 \\
&= \frac{1}{2\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\|(\mathbf{v}_i - a_{m\hat{j}} - \bar{\omega}_{m\hat{j}}\mathbf{n}_{\hat{j}} - \bar{\omega}_{m\hat{j}}\mu_v)\|^2 \\
&\quad - \frac{2}{2\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}(\mathbf{v}_i - a_{m\hat{j}} - \bar{\omega}_{m\hat{j}}\mathbf{n}_{\hat{j}} - \bar{\omega}_{m\hat{j}}\mu_v)^T\mathbf{R}_{\hat{j}}(\bar{\omega}_{m\hat{j}}\mathbf{y}_m^{-}\bar{\omega}_{m\hat{j}}\mathbf{n}_{\hat{j}} - \bar{\omega}_{m\hat{j}}\mu_y) \\
&\quad + \frac{1}{2\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\|\mathbf{R}_{\hat{j}}(\bar{\omega}_{m\hat{j}}\mathbf{y}_m - \bar{\omega}_{m\hat{j}}\mathbf{n}_{\hat{j}} - \bar{\omega}_{m\hat{j}}\mu_y)\|^2
\end{aligned} \tag{6}
$$

Now we extract the third term of Eq. (6).

$$
\begin{aligned}
&\frac{1}{2\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\|\mathbf{R}_{\hat{j}}(\bar{\omega}_{m\hat{j}}\mathbf{y}_m - \bar{\omega}_{m\hat{j}}\mathbf{n}_{\hat{j}} - \bar{\omega}_{m\hat{j}}\mu_y)\|^2 \\
&= \frac{1}{2\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\bar{\omega}_{m\hat{j}}^2\|\mathbf{R}_{\hat{j}}(\mathbf{y}_m - \mathbf{n}_{\hat{j}} - \mu_y)\|^2 \\
&= \frac{1}{2\sigma^2}\sum_{i=1}^{N} tr\Big(diag(\mathbf{P}_{:,i})diag(\mathbf{\Omega}^2_{:,\hat{j}})(\mathbf{R}_{\hat{j}}\mathbf{Y}_1)^T(\mathbf{R}_{\hat{j}}\mathbf{Y}_1)\Big) \\
&= \frac{1}{2\sigma^2}\sum_{i=1}^{N} tr\Big(diag(\mathbf{P}_{:,i})diag(\mathbf{\Omega}^2_{:,\hat{j}})\mathbf{Y}_1^T\mathbf{R}_{\hat{j}}^T\mathbf{R}_{\hat{j}}\mathbf{Y}_1\Big)
\end{aligned} \tag{7}
$$

Here, $\mathbf{Y}_1$ is a $D \times M$ matrix with columns $(\mathbf{y}_m - \mathbf{n}_{\hat{j}} - \mu_y)$. By considering the orthogonal constraint ($\mathbf{R}_{\hat{j}}^T\mathbf{R}_{\hat{j}} = \mathbf{I}$, $\mathbf{I}$ is an identity matrix), we can obtain $\frac{1}{2\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\|\mathbf{R}_{\hat{j}}(\bar{\omega}_{m\hat{j}}\mathbf{y}_m - \bar{\omega}_{m\hat{j}}\mathbf{n}_{\hat{j}} - \bar{\omega}_{m\hat{j}}\mu_y)\|^2 = \frac{1}{2\sigma^2}\sum_{i=1}^{N} tr\Big(diag(\mathbf{P}_{:,i})diag(\mathbf{\Omega}^2_{:,\hat{j}})$ $\mathbf{Y}_1^T\mathbf{Y}_1\Big)$. $tr()$ is the trace operation. Therefore, the first and third terms in Eq. (6) are constants, and E_1 can be rewritten as:

$$
\begin{aligned}
E_1 &= -\frac{1}{\sigma^2}\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}(\mathbf{v}_i - \mathbf{a}_{m\hat{j}} - \bar{\omega}_{m\hat{j}}\mathbf{n}_{\hat{j}} - \bar{\omega}_{m\hat{j}}\mu_v)^T\mathbf{R}_{\hat{j}}(\bar{\omega}_{m\hat{j}}\mathbf{y}_m - \bar{\omega}_{m\hat{j}}\mathbf{n}_{\hat{j}} - \bar{\omega}_{m\hat{j}}\mu_y) + z_1 \\
&= -\frac{1}{\sigma^2}\Bigg(\sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\bar{\omega}_{m\hat{j}}\mathbf{v}_i^T\mathbf{R}_{\hat{j}}(\mathbf{y}_m - \mathbf{n}_{\hat{j}} - \mu_y) - \sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\bar{\omega}_{m\hat{j}}\mathbf{a}_{m\hat{j}}^T\mathbf{R}_{\hat{j}}(\mathbf{y}_m - \mathbf{n}_{\hat{j}} - \mu_y) \\
&\quad - \sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\bar{\omega}_{m\hat{j}}\bar{\omega}_{m\hat{j}}\mathbf{n}_{\hat{j}}^T\mathbf{R}_{\hat{j}}(\mathbf{y}_m - \mathbf{n}_{\hat{j}} - \mu_y) - \sum_{i=1}^{N}\sum_{m=1}^{M} p_{mi}\bar{\omega}_{m\hat{j}}\bar{\omega}_{m\hat{j}}\mu_v^T\mathbf{R}_{\hat{j}}(\mathbf{y}_m - \mathbf{n}_{\hat{j}} - \mu_y)\Bigg) + z_1 \\
&= -tr\Big(\frac{1}{\sigma^2}\mathbf{Y}_1[diag(\mathbf{\Omega}_{:,\hat{j}})\mathbf{P}\mathbf{V}^T - diag(\mathbf{P1})diag(\mathbf{\Omega}_{:,\hat{j}})\mathbf{A}_{\hat{j}}^T - diag(\mathbf{\Omega}^2_{:,\hat{j}})\mathbf{P1}\mathbf{n}_{\hat{j}}^T - diag(\mathbf{\Omega}^2_{:,\hat{j}})\mathbf{P1}\mu_v^T]\mathbf{R}_{\hat{j}}\Big) + z_1
\end{aligned} \tag{8}
$$

Similarly, we can rewrite the second and third terms in Eq. (5) as follows.

$$
E_2 = tr\Big(\beta_{\text{smooth}}\sum_{\mathbf{n}_k \in N(\mathbf{n}_{\hat{j}})}[(\mathbf{n}_k - \mathbf{n}_{\hat{j}} - \mu_y)(\mathbf{n}_{\hat{j}} + \mu_v - \mathbf{n}_k - \mathbf{T}_k)^T + \mu_y(\mathbf{R}_k(\mathbf{n}_{\hat{j}} - \mathbf{n}_k) + \mathbf{n}_k + \mathbf{T}_k - \mathbf{n}_{\hat{j}} - \mu_v)^T]\mathbf{R}_{\hat{j}}\Big) + z_2 \tag{9}
$$

$$
E_3 = -tr\Big(\beta_{\text{small}}[(\mathbf{R}_{\hat{j}}^{pre})^T + \mu_y(\mu_v - \mathbf{T}_{\hat{j}}^{pre})^T]\mathbf{R}_{\hat{j}}\Big) + z_3 \tag{10}
$$

Finally, E can be rewritten as follows.

$$
E = -tr(\mathbf{H}\mathbf{R}_{\hat{j}}) + z, \tag{11}
$$

where z is a constant, and $\mathbf{H} = \mathbf{H}_1 - \mathbf{H}_2 + \mathbf{H}_3$. $\mathbf{H}_1 = \frac{1}{\sigma^2}\mathbf{Y}_1[diag(\mathbf{\Omega}_{:,\hat{j}})\mathbf{P}\mathbf{V}^T - diag(\mathbf{P1})diag(\mathbf{\Omega}_{:,\hat{j}})\mathbf{A}_{\hat{j}}^T - diag(\mathbf{\Omega}^2_{:,\hat{j}})\mathbf{P1n}_{\hat{j}}^T - diag(\mathbf{\Omega}^2_{:,\hat{j}})\mathbf{P1}\mu_v^T]$, $\mathbf{H}_2 = \beta_{\text{smooth}}\sum_{\mathbf{n}_k\in N(\mathbf{n}_{\hat{j}})}[(\mathbf{n}_k - \mathbf{n}_{\hat{j}} - \mu_y)(\mathbf{n}_{\hat{j}} + \mu_v - \mathbf{n}_k - \mathbf{T}_k)^T + \mu_y(\mathbf{R}_k(\mathbf{n}_{\hat{j}} - \mathbf{n}_k) + \mathbf{n}_k + \mathbf{T}_k - \mathbf{n}_{\hat{j}} - \mu_v)^T]$, and $\mathbf{H}_3 = \beta_{\text{small}}[(\mathbf{R}^{pre}_{\hat{j}})^T + \mu_y(\mu_v - \mathbf{T}^{pre}_{\hat{j}})^T]$.

After achieving the above function (Eq. (11)), $\mathbf{R}_{\hat{j}}$ can be then solved according to Section 4.3 (Other Constraints and Minimization) in our paper. $\mathbf{T}_{\hat{j}}$ can also be easily computed via Eq. (4) after solving $\mathbf{R}_{\hat{j}}$.

2 Bone Transformations Optimization

The transformation optimization of a specific bone is similar to the above. However, it is more complicated than the node transformation optimization since it involves multiple classes of parameters. According to our optimization scheme, the other parameters are fixed when optimizing one class of parameters. Regarding bone transformations ($\{\mathbf{R}^t_j, \mathbf{T}^t_j\}$), we optimize one bone by fixing the remaining bones. Eq. (12) shows the total energy of a specific bone $\hat{j}$ in the M-step of Section 6 in our paper. For clarity purposes, we list the key variables involved in bone transformations optimization in Table 3.

Table 3: Key variables involved in the transformation optimization of a specific bone $\hat{j}$.

Variables	Meaning	Variables	Meaning
$w_{m\hat{j}}$	weight imposed on the m-th point by bone $\hat{j}$	$\mathbf{R}^t_{\hat{j}}$	rotation of bone $\hat{j}$
$\mathbf{T}^t_{\hat{j}}$	translation of bone $\hat{j}$	$<\hat{j},k>\in\mathbb{S}$	bones $\hat{j}$ and k are connected in cluster graph $\mathbb{S}$
$\mathbf{c}_{\hat{j}k}$	location of joint by bones $\hat{j}$ and k	ζ	weight of the registration term
α	weight of the joint term	γ	weight of the small motion term

$$\begin{aligned}\mathbb{E}(\mathbf{R}^t_{\hat{j}}, \mathbf{T}^t_{\hat{j}}) =& \frac{1}{2\tau^t}\sum_{i=1}^{N^t}\sum_{m=1}^{M} p^t_{mi}\|\mathbf{v}^t_i - \mathbf{u}^t_{m\hat{j}} - w_{m\hat{j}}(\mathbf{R}^t_{\hat{j}}\mathbf{q}_m + \mathbf{T}^t_{\hat{j}})\|^2 \\ &+ \frac{\zeta}{2}\sum_{m=1}^{M}\|\mathbf{y}^t_m - \mathbf{u}^t_{m\hat{j}} - w_{m\hat{j}}(\mathbf{R}^t_{\hat{j}}\mathbf{q}_m + \mathbf{T}^t_{\hat{j}})\|^2 \\ &+ \frac{\alpha}{2}\sum_{<\hat{j},k>\in\mathbb{S}}\|(\mathbf{R}^t_{\hat{j}}\mathbf{c}_{\hat{j}k} + \mathbf{T}^t_{\hat{j}}) - (\mathbf{R}^t_k\mathbf{c}_{\hat{j}k} + \mathbf{T}^t_k)\|^2 \\ &+ \frac{\gamma}{2}\Big(\|\mathbf{R}^t_{\hat{j}} - \mathbf{R}^{pre}_{\hat{j}}\|^2_F + \|\mathbf{T}^t_{\hat{j}} - \mathbf{T}^{pre}_{\hat{j}}\|^2\Big)\end{aligned} \tag{12}$$

Here, $\mathbf{u}^t_{m\hat{j}} = \sum_{j=1,j\neq\hat{j}}^{B} w_{mj}(\mathbf{R}^t_j\mathbf{q}_m + \mathbf{T}^t_j)$ and $\mathbf{U}^t_{\hat{j}} = \{\mathbf{u}^t_{m\hat{j}}\}$.

2.1 The Relation between $\mathbf{R}^t_{\hat{j}}$ and $\mathbf{T}^t_{\hat{j}}$

Taking the partial derivative of $\mathbb{E}(\mathbf{R}^t_{\hat{j}}, \mathbf{T}^t_{\hat{j}})$ with respect to $\mathbf{T}^t_{\hat{j}}$, we can get:

$$\begin{aligned}\frac{\partial\mathbb{E}}{\partial\mathbf{T}^t_{\hat{j}}} =& -\frac{1}{\tau^t}\sum_{i=1}^{N^t}\sum_{m=1}^{M} p^t_{mi}w_{m\hat{j}}\Big(\mathbf{v}^t_i - \mathbf{u}^t_{m\hat{j}} - w_{m\hat{j}}(\mathbf{R}^t_{\hat{j}}\mathbf{q}_m + \mathbf{T}^t_{\hat{j}})\Big) \\ &- \zeta\sum_{m=1}^{M} w_{m\hat{j}}\Big(\mathbf{y}^t_m - \mathbf{u}^t_{m\hat{j}} - w_{m\hat{j}}(\mathbf{R}^t_{\hat{j}}\mathbf{q}_m + \mathbf{T}^t_{\hat{j}})\Big) \\ &+ \alpha\sum_{<\hat{j},k>\in\mathbb{S}}\Big((\mathbf{R}^t_{\hat{j}}\mathbf{c}_{\hat{j}k} + \mathbf{T}^t_{\hat{j}}) - (\mathbf{R}^t_k\mathbf{c}_{\hat{j}k} + \mathbf{T}^t_k)\Big) \\ &+ \gamma(\mathbf{T}^t_{\hat{j}} - \mathbf{T}^{pre}_{\hat{j}})\end{aligned} \tag{13}$$

Reorganizing Eq. (13) and equating it to $\mathbf{0}$, we can obtain:

$$\begin{aligned}\mathbf{T}^t_{\hat{j}}\Big(\frac{1}{\tau^t}\sum_{i=1}^{N^t}\sum_{m=1}^{M} p^t_{mi}w^2_{m\hat{j}} + \zeta\sum_{m=1}^{M} w^2_{m\hat{j}} + \alpha\sum_{<\hat{j},k>\in\mathbb{S}} 1 + \gamma\Big) =& -\mathbf{R}^t_{\hat{j}}\Big(\frac{1}{\tau^t}\sum_{i=1}^{N^t}\sum_{m=1}^{M} p^t_{mi}w^2_{m\hat{j}}\mathbf{q}_m + \zeta\sum_{m=1}^{M} w^2_{m\hat{j}}\mathbf{q}_m + \alpha\sum_{<\hat{j},k>\in\mathbb{S}}\mathbf{c}_{\hat{j}k}\Big) \\ &+ \Big(\frac{1}{\tau^t}\sum_{i=1}^{N^t}\sum_{m=1}^{M} p^t_{mi}w_{m\hat{j}}(\mathbf{v}^t_i - \mathbf{u}^t_{m\hat{j}}) + \zeta\sum_{m=1}^{M} w_{m\hat{j}}(\mathbf{y}^t_m - \mathbf{u}^t_{m\hat{j}}) + \alpha\sum_{<\hat{j},k>\in\mathbb{S}}(\mathbf{R}^t_k\mathbf{c}_{\hat{j}k} + \mathbf{T}^t_k) + \gamma\mathbf{T}^{pre}_{\hat{j}}\Big)\end{aligned} \tag{14}$$

Let $C^t_{\hat{j}} = \frac{1}{\tau^t}\sum_{i=1}^{N^t}\sum_{m=1}^{M} p^t_{mi} w^2_{m\hat{j}} + \zeta\sum_{m=1}^{M} w^2_{m\hat{j}} + \alpha\sum_{<\hat{j},k>\in\mathbb{S}} 1 + \gamma$, $\mu^t_{u\hat{j}} = \frac{1}{C^t_{\hat{j}}}\Big(\frac{1}{\tau^t}\sum_{i=1}^{N^t}\sum_{m=1}^{M} p^t_{mi} w_{m\hat{j}}(\mathbf{v}^t_i - \mathbf{u}^t_{m\hat{j}}) + \zeta\sum_{m=1}^{M} w_{m\hat{j}}(\mathbf{y}^t_m - \mathbf{u}^t_{m\hat{j}}) + \alpha\sum_{<\hat{j},k>\in\mathbb{S}}(\mathbf{R}^t_k\mathbf{c}_{\hat{j}k} + \mathbf{T}^t_k) + \gamma\mathbf{T}^{pre}_{\hat{j}}\Big)$ and $\mu^t_{q\hat{j}} = \frac{1}{C^t_{\hat{j}}}\Big(\frac{1}{\tau^t}\sum_{i=1}^{N^t}\sum_{m=1}^{M} p^t_{mi} w^2_{m\hat{j}}\mathbf{q}_m + \zeta\sum_{m=1}^{M} w^2_{m\hat{j}}\mathbf{q}_m + \alpha\sum_{<\hat{j},k>\in\mathbb{S}}\mathbf{c}_{\hat{j}k}\Big)$. Then we can obtain the relation between $\mathbf{R}^t_{\hat{j}}$ and $\mathbf{T}^t_{\hat{j}}$.

$$\mathbf{T}^t_{\hat{j}} = \mu^t_{u\hat{j}} - \mathbf{R}^t_{\hat{j}}\mu^t_{q\hat{j}} \tag{15}$$

Rewriting $C^t_{\hat{j}}$, $\mu^t_{u\hat{j}}$ and $\mu^t_{q\hat{j}}$ in the matrix form, we can get $C^t_{\hat{j}} = \frac{1}{\tau^t}(\mathbf{P}^t\mathbf{1})^T\mathbf{W}^2_{:,\hat{j}} + \zeta(\mathbf{W}^2_{:,\hat{j}})^T\mathbf{1} + \alpha\sum_{<\hat{j},k>\in\mathbb{S}} 1 + \gamma$, $\mu^t_{u\hat{j}} = \frac{1}{C^t_{\hat{j}}}\Big(\frac{1}{\tau^t}((\mathbf{P}^t\mathbf{V}^{t\,T})^T\mathbf{W}_{:,\hat{j}} - \mathbf{U}^t_{\hat{j}}diag(\mathbf{W}_{:,\hat{j}})\mathbf{P}^t\mathbf{1}) + \zeta(\mathbf{Y}^t - \mathbf{U}^t_{\hat{j}})\mathbf{W}_{:,\hat{j}} + \alpha\sum_{<\hat{j},k>\in\mathbb{S}}(\mathbf{R}^t_k\mathbf{c}_{\hat{j}k} + \mathbf{T}^t_k) + \gamma\mathbf{T}^{pre}_{\hat{j}}\Big)$, and $\mu^t_{q\hat{j}} = \frac{1}{C^t_{\hat{j}}}\Big(\frac{1}{\tau^t}\mathbf{Q}diag(\mathbf{W}^2_{:,\hat{j}})\mathbf{P}^t\mathbf{1} + \zeta\mathbf{Q}\mathbf{W}^2_{:,\hat{j}} + \alpha\sum_{<\hat{j},k>\in\mathbb{S}}\mathbf{c}_{\hat{j}k}\Big)$. $\mathbf{W}_{:,\hat{j}}$ is the $\hat{j}$-th column of $\mathbf{W}$. $\mathbf{W}^2_{:,\hat{j}}$ is the column consisting of squared values of elements in $\mathbf{W}_{:,\hat{j}}$. $\mathbf{P}^t = \{p^t_{mi}\}$. $diag()$ denotes the diagonal matrix.

2.2 Solving $\mathbf{R}^t_{\hat{j}}$

Substituting Eq. (15) into Eq. (12), we can obtain:

$$\begin{aligned}
\mathbb{E}(\mathbf{R}^t_{\hat{j}}) =& \frac{1}{2\tau^t}\sum_{i=1}^{N^t}\sum_{m=1}^{M} p^t_{mi}\|\mathbf{v}^t_i - \mathbf{u}^t_{m\hat{j}} - w_{m\hat{j}}(\mathbf{R}^t_{\hat{j}}\mathbf{q}_m + \mu^t_{u\hat{j}} - \mathbf{R}^t_{\hat{j}}\mu^t_{q\hat{j}})\|^2 \\
&+ \frac{\zeta}{2}\sum_{m=1}^{M}\|\mathbf{y}^t_m - \mathbf{u}^t_{m\hat{j}} - w_{m\hat{j}}(\mathbf{R}^t_{\hat{j}}\mathbf{q}_m + \mu^t_{u\hat{j}} - \mathbf{R}^t_{\hat{j}}\mu^t_{q\hat{j}})\|^2 \\
&+ \frac{\alpha}{2}\sum_{<\hat{j},k>\in\mathbb{S}}\|(\mathbf{R}^t_{\hat{j}}\mathbf{c}_{\hat{j}k} + \mu^t_{u\hat{j}} - \mathbf{R}^t_{\hat{j}}\mu^t_{q\hat{j}}) - (\mathbf{R}^t_k\mathbf{c}_{\hat{j}k} + \mathbf{T}^t_k)\|^2 \\
&+ \frac{\gamma}{2}\Big(\|\mathbf{R}^t_{\hat{j}} - \mathbf{R}^{pre}_{\hat{j}}\|^2_F + \|\mu^t_{u\hat{j}} - \mathbf{R}^t_{\hat{j}}\mu^t_{q\hat{j}} - \mathbf{T}^{pre}_{\hat{j}}\|^2\Big)
\end{aligned} \tag{16}$$

Now we focus on the first term on the right side and denote it as $\mathbb{E}_1$.

$$\begin{aligned}
\mathbb{E}_1 =& \frac{1}{2\tau^t}\sum_{i=1}^{N^t}\sum_{m=1}^{M} p^t_{mi}\|\mathbf{v}^t_i - \mathbf{u}^t_{m\hat{j}} - w_{m\hat{j}}(\mathbf{R}^t_{\hat{j}}\mathbf{q}_m + \mu^t_{u\hat{j}} - \mathbf{R}^t_{\hat{j}}\mu^t_{q\hat{j}})\|^2 \\
=& \frac{1}{2\tau^t}\sum_{i=1}^{N^t}\sum_{m=1}^{M} p^t_{mi}\|\mathbf{v}^t_i - \mathbf{u}^t_{m\hat{j}} - w_{m\hat{j}}\mu^t_{u\hat{j}}\|^2 \\
&- \frac{2}{2\tau^t}\sum_{i=1}^{N^t}\sum_{m=1}^{M} p^t_{mi}(\mathbf{v}^t_i - \mathbf{u}^t_{m\hat{j}} - w_{m\hat{j}}\mu^t_{u\hat{j}})^T\mathbf{R}^t_{\hat{j}}(\mathbf{q}_m w_{m\hat{j}} - \mu^t_{q\hat{j}} w_{m\hat{j}}) \\
&+ \frac{1}{2\tau^t}\sum_{i=1}^{N^t}\sum_{m=1}^{M} p^t_{mi}\|\mathbf{R}^t_{\hat{j}}(\mathbf{q}_m w_{m\hat{j}} - \mu^t_{q\hat{j}} w_{m\hat{j}})\|^2,
\end{aligned} \tag{17}$$

We extract the last term on the right side of Eq. (17) and can obtain:

$$\begin{aligned}
&\frac{1}{2\tau^t}\sum_{i=1}^{N^t}\sum_{m=1}^{M} p^t_{mi}\|\mathbf{R}^t_{\hat{j}}(\mathbf{q}_m w_{m\hat{j}} - \mu^t_{q\hat{j}} w_{m\hat{j}})\|^2 \\
&= \frac{1}{2\tau^t}\sum_{i=1}^{N^t}\sum_{m=1}^{M} p^t_{mi} w^2_{m\hat{j}}\|\mathbf{R}^t_{\hat{j}}(\mathbf{q}_m - \mu^t_{q\hat{j}})\|^2 \\
&= \frac{1}{2\tau^t}\sum_{i=1}^{N^t} tr\Big(diag(\mathbf{P}^t_{:,i})diag(\mathbf{W}^2_{:,\hat{j}})(\mathbf{R}^t_{\hat{j}}\bar{\mathbf{Q}}^t_{\hat{j}})^T(\mathbf{R}^t_{\hat{j}}\bar{\mathbf{Q}}^t_{\hat{j}})\Big) \\
&= \frac{1}{2\tau^t}\sum_{i=1}^{N^t} tr\Big(diag(\mathbf{P}^t_{:,i})diag(\mathbf{W}^2_{:,\hat{j}})(\bar{\mathbf{Q}}^t_{\hat{j}})^T\mathbf{R}^{t\,T}_{\hat{j}}\mathbf{R}^t_{\hat{j}}\bar{\mathbf{Q}}^t_{\hat{j}}\Big)
\end{aligned} \tag{18}$$

Here, $\bar{\mathbf{Q}}^t_{\hat{j}} = \mathbf{Q} - \mu^t_{q\hat{j}}\mathbf{1}$. Since $\mathbf{R}^{t\,T}_{\hat{j}}\mathbf{R}^t_{\hat{j}} = \mathbf{I}$ (the orthogonal constraint), so Eq. (18) is a constant. Thus, the first and third terms in $\mathbb{E}_1$ are constants, and $\mathbb{E}_1$ can be rewritten as:

$$
\begin{aligned}
\mathbb{E}_1 &= -\frac{1}{\tau^t}\sum_{i=1}^{N^t}\sum_{m=1}^{M} p^t_{mi}(\mathbf{v}^t_i - \mathbf{u}^t_{m\hat{j}} - w_{m\hat{j}}\mu^t_{u\hat{j}})^T \mathbf{R}^t_{\hat{j}}(\mathbf{q}_m w_{m\hat{j}} - \mu^t_{q\hat{j}} w_{m\hat{j}}) + b_1 \\
&= -\frac{1}{\tau^t} tr\Bigg(\Big(diag(\mathbf{W}_{:,\hat{j}})\mathbf{P}^t {\mathbf{V}^t}^T - diag(\mathbf{P}^t\mathbf{1})diag(\mathbf{W}_{:,\hat{j}}){\mathbf{U}^t_{\hat{j}}}^T - diag(\mathbf{W}^2_{:,\hat{j}})\mathbf{P}^t\mathbf{1}{\mu^t_{u\hat{j}}}^T\Big)\mathbf{R}^t_{\hat{j}}\bar{\mathbf{Q}}^t_{\hat{j}}\Bigg) + b_1 \\
&= -tr\Bigg(\frac{1}{\tau^t}\bar{\mathbf{Q}}^t_{\hat{j}}\Big(diag(\mathbf{W}_{:,\hat{j}})\mathbf{P}^t {\mathbf{V}^t}^T - diag(\mathbf{P}^t\mathbf{1})diag(\mathbf{W}_{:,\hat{j}}){\mathbf{U}^t_{\hat{j}}}^T - diag(\mathbf{W}^2_{:,\hat{j}})\mathbf{P}^t\mathbf{1}{\mu^t_{u\hat{j}}}^T\Big)\mathbf{R}^t_{\hat{j}}\Bigg) + b_1
\end{aligned} \tag{19}
$$

We can derive the second to fourth terms in $\mathbb{E}(\mathbf{R}^t_{\hat{j}})$ in a similar way as above.

$$
\mathbb{E}_2 = -tr\Bigg(\zeta\bar{\mathbf{Q}}^t_{\hat{j}}\Big(diag(\mathbf{W}_{:,\hat{j}})({\mathbf{Y}^t}^T - {\mathbf{U}^t_{\hat{j}}}^T - \mathbf{W}_{:,\hat{j}}{\mu^t_{u\hat{j}}}^T)\Big)\mathbf{R}^t_{\hat{j}}\Bigg) + b_2 \tag{20}
$$

$$
\mathbb{E}_3 = -tr\Big(\alpha\bar{\mathbf{C}}^t_{\hat{j}}(\tilde{\mathbf{C}}^t_{\hat{j}})^T\mathbf{R}^t_{\hat{j}}\Big) + b_3 \tag{21}
$$

$$
\mathbb{E}_4 = -tr\Big(\gamma[(\mathbf{R}^{pre}_{\hat{j}})^T + \mu^t_{q\hat{j}}(\mu^t_{u\hat{j}} - \mathbf{T}^{pre}_{\hat{j}})^T]\mathbf{R}^t_{\hat{j}}\Big) + b_3 \tag{22}
$$

Finally, $\mathbb{E}(\mathbf{R}^t_{\hat{j}})$ can be rewritten as:

$$
\mathbb{E}(\mathbf{R}^t_{\hat{j}}) = -tr(\mathbf{Z}^t_{\hat{j}}\mathbf{R}^t_{\hat{j}}) + b, \tag{23}
$$

where b is a constant, and $\mathbf{Z}^t_{\hat{j}} = \bar{\mathbf{Q}}^t_{\hat{j}}\mathbf{O}^t_{\hat{j}} + \alpha\bar{\mathbf{C}}^t_{\hat{j}}(\tilde{\mathbf{C}}^t_{\hat{j}})^T + \gamma\mathbf{H}^t_{\hat{j}}$. $\bar{\mathbf{Q}}^t_{\hat{j}} = \mathbf{Q} - \mu^t_{q\hat{j}}\mathbf{1}$, $\mathbf{O}^t_{\hat{j}} = \frac{1}{\tau^t}[diag(\mathbf{W}_{:,\hat{j}})\mathbf{P}^t{\mathbf{V}^t}^T - diag(\mathbf{P}^t\mathbf{1})diag(\mathbf{W}_{:,\hat{j}}){\mathbf{U}^t_{\hat{j}}}^T - diag(\mathbf{W}^2_{:,\hat{j}})\mathbf{P}^t\mathbf{1}{\mu^t_{u\hat{j}}}^T] + \zeta diag(\mathbf{W}_{:,\hat{j}})({\mathbf{Y}^t}^T - {\mathbf{U}^t_{\hat{j}}}^T - \mathbf{W}_{:,\hat{j}}{\mu^t_{u\hat{j}}}^T)$. $\bar{\mathbf{C}}^t_{\hat{j}}$ is a matrix with column vectors $(\mathbf{c}_{\hat{j}k} - \mu^t_{q\hat{j}})$, and $\tilde{\mathbf{C}}^t_{\hat{j}}$ is a matrix with column vectors $(\mathbf{R}^t_k\mathbf{c}_{\hat{j}k} + \mathbf{T}^t_k - \mu^t_{u\hat{j}})$. $\mathbf{H}^t_{\hat{j}} = (\mathbf{R}^{pre}_{\hat{j}})^T + \mu^t_{q\hat{j}}(\mu^t_{u\hat{j}} - \mathbf{T}^{pre}_{\hat{j}})^T$.

$\mathbf{R}^t_{\hat{j}}$ can be solved according to Section 6.4 (Parameter Estimation) in our paper. $\mathbf{T}^t_{\hat{j}}$ can be then computed via Eq. (15).